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“K-Bounded Polynomials”

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K-BOUNDED POLYNOMIALS

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Abstract

For a Banach space $E$ we define the class $\mathcal{P}_K(NE)$ of $K$-bounded $N$-homogeneous polynomials, where $K$ is a bounded subset of $E'$. We investigate properties of $K$ which relate the space $\mathcal{P}_K(NE)$ with usual subspaces of $\mathcal{P}(NE)$. We prove that $K$-bounded polynomials are approximable when $K$ is a compact set where the identity can be uniformly approximated by finite rank operators. The same is true when $K$ is contained in the absolutely convex hull of a weakly null basic sequence of $E'$. Moreover, in this case we prove that every $K$-bounded polynomial is extendible to any larger space.

1. Introduction

If $E$ is a Banach space and $K$ is a bounded subset of its dual, we say that a scalar valued $N$-homogeneous polynomial $P$ on $E$ is $K$-bounded if there is a positive constant $C$ such that the inequality $|P(x)| \leq C \sup \{ |\gamma(x)|^N : \gamma \in K \}$ holds for all $x \in E$. Note that continuity is equivalent to $B_{E'}$-boundedness, and also (see Proposition 3.2) that finite type polynomials correspond to $K$-bounded with $K$ finite.

A result of E. Toma [12] (see also [5]) states that a continuous homogeneous polynomial is weakly continuous on bounded sets if and only if it is $K$-bounded for some compact set $K$. Our interest in $K$-bounded polynomials was originally motivated by this result. The closure of the space of finite type polynomials is the space of ‘approximable’ polynomials and is a subspace of the space of polynomials which are weakly continuous on bounded sets. Thus, we set out to clarify the relationship between ‘approximable’ and ‘$K$-bounded’ (with $K$ something between ‘finite’ and ‘compact’). We obtain several sufficient conditions for approximability of a polynomial and are naturally led to consider also the problem of extendibility (to any larger Banach space) of $K$-bounded polynomials for several types of subsets $K$ of $E'$. Note that all finite type polynomials (and even all integral polynomials [6]) are extendible, but the same does not hold for approximable polynomials.

The paper is organised as follows. In section 2, we set our notation and give a few basic properties about $K$-bounded polynomials, as well as a new (isometric) version of a result of Aron and Galindo regarding the Aron-Berner extension of a $K$-bounded polynomial. Section 3 is devoted to the search for conditions on $K$ which ensure approximability and extendibility.
2. Basic properties

Throughout, $E$ will be a Banach space over the scalar field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and $E'$ will denote its dual space. The space of continuous $N$-homogeneous polynomials from $E$ into $\mathbb{K}$ will be denoted by $\mathcal{P}(N E)$. This is a Banach space with the norm $\|P\| = \sup\{\|P(x)\| : \|x\| \leq 1\}$. If $P \in \mathcal{P}(N E)$, $\check{P}$ will denote the continuous symmetric $N$-linear form associated with $P$.

A polynomial $P \in \mathcal{P}(N E)$ is said to be of finite type if there exists a finite subset $\{\varphi_i\}_{i=1}^m$ of $E'$ such that

$$P(x) = \sum_{i=1}^m \varphi_i^N(x)$$

for complex $E$. When $E$ is a real Banach space and $N$ is even, the representation must take into account the signs, so $P(x) = \sum_{i=1}^m \varphi_i^N(x) - \sum_{j=1}^n \psi_j^N(x)$. We will denote by $\mathcal{P}_f(N E)$ the space of $N$-homogeneous polynomials of finite type and its closure, in $(\mathcal{P}(N E), \|\cdot\|)$, by $\mathcal{P}_c(N E)$. Polynomials in $\mathcal{P}_c(N E)$ will be called approximable.

$\mathcal{P}_w(N E)$ will denote the space of polynomials which are weakly continuous on bounded sets. This is a closed subspace of $\mathcal{P}(N E)$, and we have

$$\mathcal{P}_f(N E) \subset \mathcal{P}_c(N E) \subset \mathcal{P}_w(N E) \subset \mathcal{P}(N E) \quad (1)$$

Let $K$ be a bounded subset of $E'$. For $x \in E$, we define

$$\|x\|_K = \sup_{\gamma \in K} |\gamma(x)|$$

which is a continuous semi-norm on $E$.

**Definition 2.1.** We say that an $N$-homogeneous polynomial $P$ is $K$-bounded if there exists a positive constant $C$ such that

$$|P(x)| \leq C \|x\|_K^N \quad (2)$$

for all $x$ in $E$. The smallest constant $C$ that verifies (2) is called $\|P\|_K$.

Since $\|\cdot\|_K$ is a continuous semi-norm on $E$, every $K$-bounded polynomial is continuous. The space of $K$-bounded $N$-homogeneous polynomials will be denoted by $\mathcal{P}_K(N E)$. On $\mathcal{P}_K(N E)$, $\|\cdot\|_K$ is a norm and $(\mathcal{P}_K(N E), \|\cdot\|_K)$ is a Banach space.

We also say that an $N$-linear form $\Phi : E^N \to \mathbb{K}$ is $K$-bounded if there exists a positive constant $C$ such that

$$|\Phi(x_1, \ldots, x_N)| \leq C \|x_1\|_K \cdots \|x_N\|_K \quad (3)$$

for all $(x_1, \ldots, x_N) \in E^N$ and $\|\Phi\|_K$ will be the smallest constant $C$ verifying (3).

Clearly, every $K$-bounded $\Phi$ is continuous. From the polarization formula, we have the following inequalities

$$\|P\|_K \leq \|\check{P}\|_K \leq \frac{N^N}{N!} \|P\|_K$$

It follows that there exists a one to one correspondence between $K$-bounded $N$-homogeneous polynomials and $K$-bounded symmetric $N$-linear forms.
Remark 2.2. For every \( x, y \in E \), and every \( P \in \mathcal{P}_K(N E) \), if \( \|x - y\|_K = 0 \), then \( P(x) = P(y) \).

\[
|P(x) - P(y)| \leq \sum_{i=1}^n |P(x, \ldots, x) - P(y, \ldots, x)| + \sum_{i=1}^n |P(y, \ldots, x) - P(y, y, \ldots, x)|
\]

\[
\leq \|\hat{P}\|_K \|x - y\|_K \|\gamma\|_K^{N-1} + \|\hat{P}\|_K \|x - y\|_K \|\gamma\|_K^{N-1}
\]

\[
\leq N \|\hat{P}\|_K \|x - y\|_K \max\{\|x\|_K, \|y\|_K\}^{N-1}.
\]

Remark 2.3. Since \( K_1 \subset K_2 \subset E' \) implies \( \|x\|_{K_1} \leq \|x\|_{K_2} \) for all \( x \in E \), every \( K_1 \)-bounded polynomial \( P \) is \( K_2 \)-bounded, with

\[
\|P\|_{K_2} \leq \|P\|_{K_1}.
\]

Also, if \( K \subset E' \) and \( \hat{K} = \Gamma(K) \) is its closed, convex, balanced hull, then \( \|x\|_{\hat{K}} = \|x\| K \) for all \( x \in E \). Indeed, let \( \gamma_0 \in \Gamma(K) \), say \( \gamma_0 = \sum_{i=1}^n \alpha_i \gamma_i \), where \( \gamma_i \in K \), \( \alpha_i \in \mathbb{K} \) for \( i = 1, \ldots, n \) and \( \sum_{i=1}^n |\alpha_i| \leq 1 \). Then, for all \( x \in E \), we have

\[
|\gamma_0(x)| = \left| \sum_{i=1}^n \alpha_i \gamma_i(x) \right| \leq \sup_{j=1}^n |\gamma_j(x)| \sum_{i=1}^n |\alpha_i| \leq \sup_{\gamma \in K} |\gamma(x)| = \|x\|_K
\]

and

\[
\|x\|_{\hat{K}} = \sup_{\gamma \in \Gamma(K)} |\gamma(x)| = \sup_{\gamma \in \Gamma(K)} |\gamma(x)| \leq \|x\|_K.
\]

From this and the fact that \( K \subset \hat{K} \), it follows that \( \|x\|_K = \|x\|_{\hat{K}} \). Therefore, \( \mathcal{P}_K(N E) = \mathcal{P}_{\hat{K}}(N E) \) with \( \|P\|_K = \|P\|_{\hat{K}} \).

Since \( \|\cdot\|_K \) is a continuous semi-norm on \( E \), then \( \circ K = \{ x \in E : \|x\|_K = 0 \} \) is a closed subspace of \( E \). On \( E/\circ K \), we can define the following norm

\[
\|\Pi(x)\| = \|x\|_K
\]

where \( \Pi : E \to E/\circ K \) is the quotient projection. The completion of \( E/\circ K \), \((E_K, \|\cdot\|)\) is a Banach space.

**Lemma 2.4.** Let \( K \) be a bounded subset of \( E' \). Then the spaces \((\mathcal{P}_K(N E), \|\cdot\|_K)\) and \((\mathcal{P}(N E_K), \|\cdot\|)\) are isometrically isomorphic.

**Proof.** For \( P \in \mathcal{P}_K(N E) \) we define \( Q : E/\circ K \to \mathbb{K} \) by

\[
Q(\Pi(x)) = P(x) \quad \forall x \in E.
\]

\( Q \) is well defined because of remark 2.2. Also, \( Q \) is an \( N \)-homogeneous polynomial and

\[
\|Q\| = \sup\{|Q(y)| : y \in E/\circ K, \|y\| = 1\} = \sup\{|Q(\Pi(x))| : x \in E, \|\Pi(x)\| = 1\}
\]

\[
= \sup\{|P(x)| : x \in E, \|x\|_K = 1\} = \|P\|_K.
\]

Thus \( Q \) is continuous and can be extended to an \( N \)-homogeneous polynomial on \( E_K \) with the same norm; this extension will still be called \( Q \).
Conversely, let $Q \in \mathcal{P}(N^E_K)$. Clearly, $P(x) = Q(\Pi(x))$ is a $K$-bounded $N$-homogeneous polynomial and $\|P\|_K = \|Q\|$. ■

It is known that every $P \in \mathcal{P}(N^E)$ extends to $E''$ via the Aron-Berner morphism [1], and that this morphism preserves norms [7]. We will denote that extension by $\mathcal{P}$. Since $K \subset E'$, it can be considered as a subset of $E''$. Aron and Galindo [3, corollary 8] proved that the Aron-Berner extension of a $K$-bounded polynomial is $K$-bounded, when $K$ is a weakly compact set. Using the construction of the preceding lemma, we will give another proof of this fact. Moreover, we will show that the Aron-Berner morphism is a $\| \cdot \|_K$-isometry.

**Proposition 2.5.** Let $K$ be a relatively weakly compact subset of $E'$. Then the Aron-Berner morphism is an isometry from $(\mathcal{P}_K(N^E), \| \cdot \|_K)$ into $(\mathcal{P}_K(N^E'), \| \cdot \|_K)$ for every $N \in \mathbb{N}$.

**Proof.** For $P \in \mathcal{P}_K(N^E)$, let $Q \in \mathcal{P}_K(N^E_K)$ as in lemma 2.4, let $\overline{Q} \in \mathcal{P}_K(N^E_K)$ be the Aron-Berner extension of $Q$ and $\mathcal{P} = \overline{Q} \circ \Pi'' : E'' \to K$ where $\Pi''$ is the bitranspose of $\Pi$. Using the characterization of the Aron-Berner extension due to Zalduendo [13, theorem 2], it is easy to check that $\mathcal{P}$ is the Aron-Berner extension of $P$. Let us see that $\mathcal{P}$ is $K$-bounded with $\|\mathcal{P}\|_K = \|P\|_K$.

For $x'' \in E''$,

$$\|\mathcal{P}(x'')\| = \|\overline{Q}(\Pi''(x''))\| \leq \|\overline{Q}\| \|\Pi''(x'')\|^{\gamma} = \|P\|_K \|\Pi''(x'')\|^{\gamma} \tag{4}$$

We claim that $\Pi''(B_{E'_K})$ is contained in $\overline{\Gamma(K)}$, the closed, convex, balanced hull of $K$. To see this, let $\beta \in B_{E'_K}$, then

$$\|\Pi''(\beta)(x)\| = \|\beta(\Pi(x))\| \leq \|\beta\| \|\Pi(x)\|^{\gamma} \leq \|x\|_K = \sup_{\gamma \in K} \gamma(x) \quad \forall x \in E$$

By the Hahn-Banach theorem, $\Pi''(\beta)$ belongs to the weak-star closure of $\overline{\Gamma(K)}$, $\overline{\Gamma(K)}^{\omega^*}$. Since $K$ is relatively weakly compact, $\overline{\Gamma(K)}$ is weakly compact. Then $\overline{\Gamma(K)}$ is weak-star compact and it follows that $\overline{\Gamma(K)}^{\omega^*} = \overline{\Gamma(K)}$. Hence, $\Pi''(B_{E'_K}) \subset \overline{\Gamma(K)}$. Returning to (4),

$$\|\mathcal{P}(x'')\| \leq \|P\|_K \sup_{\varphi \in \overline{\Gamma(K)}} |x''(\varphi)|^{\gamma} = \|P\|_K \sup_{\varphi \in K} |x''(\varphi)|^{\gamma} = \|P\|_K \|x''\|_K^{\gamma}$$

Therefore, $\mathcal{P}$ is $K$-bounded and $\|\mathcal{P}\|_K = \|P\|_K$. ■

3. Main results

We want to describe $K$-bounded polynomials corresponding to different classes of sets $K$. We begin by considering finite dimensional subsets of $E''$ which will be related to finite type polynomials as we will see in proposition 3.2. First, we need the following lemma.

**Lemma 3.1.** A polynomial $P \in \mathcal{P}(N^E)$ is of finite type if and only if its associated operator $T_P : E \to \mathcal{P}(N^E)$ has finite rank.

**Proof.** If $P(x) = \sum_{i=1}^m \varphi_i^N(x)$, then $T_P(x) = \sum_{i=1}^m \varphi(x) \varphi_i^{N-1}$ which is a finite rank operator. Conversely, suppose $T_P$ is a finite rank operator and let $\Pi : E \to E/\ker T_P$ be the quotient
projection. We define a polynomial $\tilde{P}$ on $E/\ker T_P$ by $\tilde{P}(\Pi(x)) = P(x)$. To see that $\tilde{P}$ is well defined, let $\Pi(x) = \Pi(y)$. Since $T_P(x) = T_P(y)$,

$$P(x) = T_P(x)(x) = T_P(y)(x) = \tilde{P}(y, x, x, \ldots, x) = T_P(y)(x, x, \ldots, x) = \tilde{P}(y, y, x, x, \ldots, x) = \cdots = P(y)$$

Since $E/\ker T_P$ is a finite dimensional space, $\tilde{P}$ becomes a finite type polynomial and so does $P$. ■

**Proposition 3.2.** Let $K \subset E'$ be a bounded set. Then every $K$-bounded $N$-homogeneous polynomial is of finite type if and only if the subspace spanned by $K$ is finite dimensional.

**Proof.** Suppose span$(K)$ is finite dimensional and let $\{\gamma_1, \ldots, \gamma_m\} \subset E'$ be a basis of span$(K)$ such that $K \subset \Gamma(\{\gamma_1, \ldots, \gamma_m\})$. If $u : E \to \mathbb{K}^m$ is defined by $u(x) = (\gamma_1(x), \ldots, \gamma_m(x))$, then $u$ is a continuous linear map satisfying $\|u(x)\|_\infty \geq \|x\|_K$. Given $P \in \mathcal{P}_K(N E)$, we define $Q : \text{Im}(u) \to \mathbb{K}$ by $Q(u(x)) = P(x)$, which is well defined by remark 2.2. Since $Q$ is a continuous $N$-homogeneous polynomial from a subspace of $\mathbb{K}^m$ into $\mathbb{K}$, we can write

$$Q(z) = \sum_{|\alpha| = N} a_\alpha z_1^{\alpha_1} \cdots z_m^{\alpha_m} \quad \forall z = (z_1, \ldots, z_m) \in \text{Im}(u)$$

where $a_\alpha \in \mathbb{K}$, $\alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}_0^m$, $|\alpha| = \alpha_1 + \cdots + \alpha_m$. Then,

$$P(x) = Q(u(x)) = \sum_{|\alpha| = N} a_\alpha \gamma_1(x)^{\alpha_1} \cdots \gamma_m(x)^{\alpha_m} \quad \forall x \in E.$$

In particular, $P$ is a polynomial of finite type.

To see the converse we will use the identification given in lemma 2.4. Let $P \in \mathcal{P}_K(N E)$ of finite type; then the corresponding polynomial $Q \in \mathcal{P}^{(N E)}$ is of finite type too. Indeed, $P$ being of finite type, its associated operator $T_P : E \to \mathcal{P}_K(N^{-1} E)$ has finite rank. Since $T_P = T_Q \circ \Pi$, where $T_Q : E_K \to \mathcal{P}^{(N^{-1} E)}$ is the operator associated to $Q$ and $\Pi : E \to E_K$ is the natural projection, then $T_Q$ has finite rank. By lemma 3.1, $Q$ is a polynomial of finite type and then every continuous $N$-homogeneous polynomial on $E_K$ is of finite type. We conclude that $E_K$ is finite dimensional. Thus the subspace spanned by $K$ has finite dimension. ■

As a corollary, we have that a $K$-bounded polynomial of finite type can be written in terms of $K$-bounded functionals (just compose with $\Pi$ the functionals representing $Q$ as a finite type polynomial).

It is clear that every polynomial of finite type is $K$-bounded for a finite set $K$, so we have

$$\mathcal{P}_f(N E) = \bigcup_{K \subset E' \text{ finite}} \mathcal{P}_K(N E).$$

In [12, 5] it is shown that

$$\mathcal{P}_w(N E) = \bigcup_{K \subset E' \text{ compact}} \mathcal{P}_K(N E).$$

and clearly

$$\mathcal{P}(N E) = \mathcal{P}_{B_{E'}}(N E).$$
where $B_{E'}$ denotes the closed unit ball of $E'$.

Taking into account the inclusions given in (1), we will try to find sets $K \subset E'$ for which $K$-bounded polynomials are approximable.

Since approximable polynomials are $w$-continuous on bounded sets, we start by considering compact subsets $K$ of $E'$. In addition, $w$-continuous polynomials on bounded sets are approximable when $E'$ has the approximation property. This suggests the following proposition:

**Proposition 3.3.** Let $K \subset E'$ be a compact set such that the identity $\text{Id} : E' \to E'$ can be uniformly approximated on $K$ by finite rank operators. Then every $K$-bounded $N$-homogeneous polynomial is approximable.

**Proof.** Without loss of generality, we suppose $K \subset B_{E'}$. Let $P \in \mathcal{P}_K(N,E)$ and $dP \in \mathcal{P}(N^{-1}E;E')$ its derivative. We have

$$
E \xrightarrow{dP} E' \\
P \downarrow \\
E_K \xrightarrow{dQ} E'_K
$$

Note that

$$
dP(B_E) = \Pi'(dQ(\Pi(B_E))) \subset \Pi'(dQ(B_{E_K})) \subset \|dQ\|\Pi'(B_{E_K}) \subset \|dQ\|\Gamma(K)
$$

(the last inclusion was explained in proposition 2.5). Furthermore, $K_1 = \|dQ\|\Gamma(K)$ is a compact subset of $E'$ on which the identity can also be uniformly approximated by finite rank operators. Thus, for each $n \in \mathbb{N}$, there exists a finite rank operator $I_n : E' \to E'$ verifying

$$
\|I_n(\gamma) - \gamma\| \leq \frac{1}{n} \quad \forall \gamma \in K_1
$$

so

$$
\|I_n(dP(x)) - dP(x)\| \leq \frac{1}{n} \quad \forall x \in B_E.
$$

We define $P_n(x) = \frac{1}{N}I_n(dP(x))(x)$. Since $dP_n = I_n \circ dP$ and $T_{P_n}(x)(y) = \frac{1}{N}(dP_n(y))(x)$ then $T_{P_n}$ has finite rank which implies, by lemma 3.1, that $P_n$ is a polynomial of finite type. We also have

$$
|P_n(x) - P(x)| = \left| \frac{1}{N}(I_n(dP(x))(x) - \frac{1}{N}dP(x))(x) \right| \leq \frac{1}{N} \frac{1}{n} \quad \forall x \in B_E.
$$

Therefore, $P$ is approximable. □

It is known (see, for example, [11]) that a subset $K$ of $E'$ is compact if and only if it is contained in the closed convex balanced hull of a null sequence. By remark 2.3,

$$
\bigcup_{K \subset E', K \text{ compact}} \mathcal{P}_K(N,E) = \bigcup_{K = \{\gamma_n\}_{n \in \mathbb{N}} \subset E', \|\gamma_n\| \to 0} \mathcal{P}_K(N,E).
$$

Let us consider $K = \{\gamma_n\}_{n \in \mathbb{N}} \subset E'$ where $\|\gamma_n\| \to 0$. We can define a linear operator $u : E \to c_0$ by

$$
u(x) = (\gamma_1(x), \gamma_2(x), \ldots, \gamma_n(x), \ldots) \quad (5)
$$

which is easily seen to be compact. This fact will enable us to prove that $K$-bounded polynomials are approximable, given some further assumption on the image of $u$. 
Proposition 3.4. Let \( K = \{ \gamma_n \}_{n \in \mathbb{N}} \subset E' \) where \( \| \gamma_n \| \longrightarrow 0 \) and let \( u \) as in (5). If \( \overline{\text{Im}(u)} \) has the approximation property, then \( K \)-bounded homogeneous polynomials are approximable.

Proof. Let \( N \in \mathbb{N} \), and \( P \in \mathcal{P}_K(N) \). We define \( Q : \text{Im}(u) \rightarrow \mathbb{K} \) by \( Q(u(x)) = P(x) \), which again is well defined by remark 2.2 and is a continuous \( N \)-homogeneous polynomial with

\[
\| Q \| = \sup\{ |Q(u(x))| : \| u(x) \|_{\infty} \leq 1 \} = \sup\{ |P(x)| : \| x \|_K \leq 1 \} = \| P \|_K
\]

We can extend \( Q \) to a continuous \( N \)-homogeneous polynomial on \( \overline{\text{Im}(u)} \) with the same norm, which will still be called \( Q \). This gives us the following diagram

\[
\xymatrix{ u \ar[r]^{\text{Im}(u)} & \mathbb{K} \\
E \ar[u]^{\overline{u}} \ar[r]_{P} & \mathbb{K} \ar[u]_{Q}
}
\]

Since \( \overline{u(B_E)} \subset \overline{\text{Im}(u)} \), \( u \) is compact and \( \overline{\text{Im}(u)} \) has the approximation property, there exist finite rank operators \( T_n : \overline{\text{Im}(u)} \rightarrow \overline{\text{Im}(u)} \) such that

\[
\| T_n(u(x)) - u(x) \|_{\infty} < \frac{1}{n} \quad \forall x \in B_E
\]

In this way we have a sequence of finite type polynomials \( \{ P_n \}_{n \in \mathbb{N}} \subset \mathcal{P}_f(N) \), given by \( P_n(x) = (Q \circ T_n \circ u)(x) \), approximating \( P \). Indeed,

\[
|P(x) - P_n(x)| = |Q(u(x)) - Q(T_n(u(x)))| \leq M \| u(x) - T_n(u(x)) \|_{\infty} \leq M \frac{1}{n} \quad \forall x \in B_E
\]

where the constant \( M \) can be chosen independent of \( x \in B_E \) and \( n \in \mathbb{N} \) (see remark 2.2).

As a consequence of this proposition we derive the following result that Grothendieck proved (see [11]) while studying the existence of spaces without the approximation property:

If there exists a Banach space without the approximation property then there exists a subspace of \( c_0 \) without the approximation property.

To see this, let \( X \) be a Banach space without the approximation property. We proceed as in [2]. There is a Banach space \( Z \) and a compact operator \( T : Z \rightarrow X \) which is not approximable by finite rank operators. If \( Y = Z \oplus X' \), we define \( S : Y \rightarrow Y' \) by \( S(z, x') = (T'x', Tz) \), where \( z \in Z \), \( x' \in X' \) and \( T' \) is the transpose of \( T \). Thus \( S \) is a compact operator that cannot be approximated by finite rank operators. Indeed, the existence of finite rank operators approximating \( S \) from \( Y \) into \( Y' \) would imply the existence of finite rank operators from \( Z \) into \( X'' \) approximating \( T \). \( T \) being compact, it would be possible to construct finite rank operators from \( Z \) into \( X \) approximating \( T \) (see [11, lemma 1.e.6]) and that is absurd. Now, by means of the compacity of \( S \), the 2-homogeneous polynomial \( P \in \mathcal{P}(2Y) \), \( P(y) = S(y)I(y) \), is \( w \)-continuous on bounded sets [4] but is not approximable. Therefore, \( P \) is \( K \)-bounded, for some \( K = \{ \gamma_n \}_{n \in \mathbb{N}} \subset Y' \), where \( \| \gamma_n \| \rightarrow 0 \), and defining \( u \) as in (5), the subspace \( \overline{\text{Im}(u)} \) of \( c_0 \) fails to have the approximation property.

From the proof above, we can conclude that the existence of a Banach space without the approximation property is equivalent to the existence of a homogeneous non-approximable polynomial which is \( w \)-continuous on bounded sets.

There is no general method to decide whether a Banach space has the approximation property or not. However, every Hilbert space has it. Let \( K = \{ \gamma_n \}_{n \in \mathbb{N}} \subset E' \), where \( \sum_{n=1}^{\infty} \| \gamma_n \|^2 < \infty \). Now,
we can modify the construction (5) by putting \( u : E \to \ell_2, u(x) = (\gamma_1(x), \ldots, \gamma_n(x), \ldots) \), which is also a compact operator. Defining the polynomial \( Q : Im(u) \subset \ell_2 \to IK \) by \( Q(u(x)) = P(x) \) we note that in this case
\[
|Q(u(x))| \leq \|P\|_K \|u(x)\|_\infty^N \leq \|P\|_K \|u(x)\|_2^N
\]

Since \( \overline{Im(u)} \) is a Hilbert space, we can proceed as in proposition 3.4 to conclude that every \( K \)-bounded polynomial is approximable. Moreover, working on a Hilbert space we can state an extension result. We have:

**Proposition 3.5.** Let \( K = \{ \gamma_n \}_{n \in IN} \subset E' \) such that \( \sum_{n=1}^\infty \|\gamma_n\|^2 < \infty \). Then every \( K \)-bounded polynomial \( P \in \mathcal{P}_K(N E) \) is approximable. Moreover, if \( G \) is a Banach space containing \( E \), there exists \( \bar{P} \in \mathcal{P}(NG) \) which is an extension of \( P \).

**Proof.** We prove the second statement, the first one having been explained in the previous paragraph. If \( E \subset G \), for each \( n \in IN \), we have an extension of \( \gamma_n \), \( \bar{\gamma}_n \in G' \), with the same norm. As \( \sum_{n=1}^\infty \|\bar{\gamma}_n\|^2 < \infty \), the operator \( u : G \to \ell_2, \bar{u}(x) = (\bar{\gamma}_1(x), \ldots, \bar{\gamma}_n(x), \ldots) \) is an extension of \( u \).

Let \( Q : Im(u) \to IK, \bar{Q}(y) = Q(\Pi(y)) \), where \( \Pi \) is the orthogonal projection onto \( \overline{Im(u)} \). This completes the following diagram:

\[
\begin{array}{ccccc}
G & \xrightarrow{u} & \overline{Im(u)} & \xrightarrow{Q} & IK \\
\uparrow i & & \downarrow \Pi & & \\
E & \xrightarrow{u} & \overline{Im(u)} & \xrightarrow{Q} & IK
\end{array}
\]

We may define \( \bar{P} : G \to IK, \bar{P}(x) = \bar{Q}(\bar{u}(x)). \) Thus \( \bar{P} \in \mathcal{P}(NG) \) and becomes an extension of \( P \). \( \blacksquare \)

In [10], Kirwan and Ryan gave the following definition:

**Definition 3.6.** A polynomial \( P \in \mathcal{P}(NE) \) will be called **extendible** if, for every Banach space \( G \supset E \), there exists \( \bar{P} \in \mathcal{P}(NG) \) which is an extension of \( P \).

Proposition 3.5 states that if we consider \( K = \{ \gamma_n \}_{n \in IN} \subset E' \) with \( \sum_{n=1}^\infty \|\gamma_n\|^2 < \infty \), then every \( K \)-bounded polynomial is extendible. Moreover, for each \( G \supset E \), there exists an extension morphism
\[
\Lambda : \mathcal{P}_K(NE) \longrightarrow \mathcal{P}(NG) \\
P \longrightarrow Q \circ \Pi \circ \bar{u}
\]

We will now investigate those bounded subsets \( K \) of \( E' \) for which every \( K \)-bounded polynomial is extendible and whether the resulting extension becomes \( \bar{K} \)-bounded, where \( \bar{K} = \{ \bar{\gamma} : \gamma \in K \} \) and \( \bar{\gamma} \) is a norm preserving extension of \( \gamma \) to \( G \). Moreover, we want to establish conditions for the existence of an extension morphism. Still, we keep in mind the problem of approximability of \( K \)-bounded polynomials.

Note that there are non-extendible approximable polynomials. For instance, \( P \in \mathcal{P}(2\ell_2) \), given by \( P(x) = \sum_{n=1}^\infty \frac{x_n^2}{n} \), is approximable but not nuclear, so it cannot be extendible [10, proposition 8].
Proposition 3.7. Let \( \{\gamma_n\}_{n \in \mathbb{N}} \subset E' \) be a basic sequence such that \( \gamma_n \xrightarrow{w} 0 \) and let \( K = \{\gamma_n\}_{n \in \mathbb{N}} \). If \( P \in \mathcal{P}_K^N(E) \) then \( P \) is approximable and extendible. Furthermore, if \( E \subset G \), then there exists an extension morphism \( \Lambda : \mathcal{P}_K^N(E) \to \mathcal{P}_{\tilde{K}}^N(G) \), where \( \tilde{K} = \{\tilde{\gamma}_n\}_{n \in \mathbb{N}} \), with \( \tilde{\gamma}_n \) a norm preserving extension of \( \gamma_n \). The morphism is an isometry if we consider the \( \|\cdot\|_K \) and \( \|\cdot\|_{\tilde{K}} \) norms.

**Proof.** As in (5), let \( u : E \to c_0 \), with \( u(x) = (\gamma_1(x), \ldots, \gamma_n(x), \ldots) \). Since \( \{\|\gamma_n\|\}_{n \in \mathbb{N}} \) is bounded, \( u \in \mathcal{L}(E; c_0) \) and \( \|u(x)\| = \|x\|_K \). Let \( \pi : E'' \to c_0 \), \( \pi(z) = (z(\gamma_1), \ldots, z(\gamma_n), \ldots) \). Again, \( \pi \in \mathcal{L}(E''; c_0) \) and \( \|\pi(z)\| = \|z\|_K \), for all \( z \in E'' \).

Since \( \{\gamma_n\}_{n \in \mathbb{N}} \) is a basic sequence in \( E' \), there exists a sequence \( \{z_n\}_{n \in \mathbb{N}} \subset E'' \) such that \( z_n(\gamma_m) = \delta_{nm} \) and so \( \pi(z_n) = e_n \), where \( \{e_n\}_{n \in \mathbb{N}} \) is the unit vector basis of \( c_0 \). It follows that \( Im(\pi) \) is a dense subspace of \( c_0 \).

Let \( P \in \mathcal{P}_K^N(E) \). As stated in proposition 2.5, its Aron-Berner extension \( \overline{\mathcal{P}} \) belongs to \( \mathcal{P}_K^N(E'') \). Consider \( Q : Im(\pi) \to IK \) defined by \( Q(\pi(z)) = \overline{\mathcal{P}}(z) \). \( Q \) is a continuous \( N \)-homogeneous polynomial with \( \|Q\| = \|\overline{\mathcal{P}}\|_K = \|P\|_K \). We may extend \( Q \) to a continuous \( N \)-homogeneous polynomial on \( c_0 \), which will still be called \( Q \), with the same norm. This gives us the following diagram:

\[
\begin{array}{ccc}
E'' & \xrightarrow{\pi} & c_0 \\
\uparrow & & \uparrow \\
E & \xrightarrow{P} & IK \\
\end{array}
\]

Being a continuous \( N \)-homogeneous polynomial on \( c_0 \), \( Q \) admits a representation as in (6).

Thus

\[
P = \sum_{(i_1, \ldots, i_N) \in D} a_{i_1, \ldots, i_N} \gamma_{i_1} \cdots \gamma_{i_N}
\]

It can be seen that the series (7) converges in the norms of both \( \mathcal{P}^N(E) \) and \( \mathcal{P}_K^N(E) \). Moreover, from the isometric isomorphism between \( \mathcal{P}_K^N(E) \) and \( \mathcal{P}(^Nc_0) \) it follows that the sequence \( \{\gamma_{i_1} \cdots \gamma_{i_N} \}_{(i_1, \ldots, i_N) \in D} \) (with the square ordering) is a Schauder basis of \( (\mathcal{P}_K^N(E), \|\cdot\|_K) \). Now that we have proved that \( P \) is approximable, we turn to the extension morphism.

Let \( G \supset E \) and \( \overline{u} : G \to \ell_\infty \) given by \( \overline{u}(y) = (\overline{\gamma}_1(y), \ldots, \overline{\gamma}_n(y), \ldots) \), with \( \overline{\gamma}_n \in G' \) a norm preserving extension of \( \gamma_n \). Let us consider \( \overline{Q} \in \mathcal{P}^N(\ell_\infty) \) the Aron-Berner extension of \( Q \in \mathcal{P}(^Nc_0) \).
Then we have the following diagram:

\[
\begin{array}{cccc}
G & \xrightarrow{u} & \ell_\infty & \\
E & \xrightarrow{u} & c_0 & \xrightarrow{Q} IK \\
\end{array}
\]

where \( J \) is the canonical inclusion from \( c_0 \) into \( \ell_\infty \).

If we define \( \tilde{P} : G \rightarrow IK \) by \( \tilde{P}(y) = Q(\tilde{u}(y)) \) we obtain an extension of \( P \) to \( G \) with

\[
|\tilde{P}(y)| = |Q(\tilde{u}(y))| \leq \|Q\| |\tilde{u}(y)|^N = \|P\|_K \|y\|^N \quad \forall y \in G
\]

This implies that \( \tilde{P} \) is \( \tilde{K} \)-bounded and \( \|\tilde{P}\|_{\tilde{K}} = \|P\|_K \). Hence, the extension morphism

\[
\Lambda : \mathcal{P}_K(^N_E) \rightarrow \mathcal{P}_{\tilde{K}}(^{N'}_G)
\]

\[
P \mapsto Q \circ \tilde{u}
\]

is an isometry. ■

Remark 3.8. Note that, in the previous proposition, \( K \) is a weakly null sequence, although we are mainly concerned with norm null sequences. In the norm null case, the result is stronger in the sense that \( \tilde{K} \) turns out to be a norm null sequence too. In the general case, when \( \gamma_n \xrightarrow{w} 0 \), it is possible to choose \( \tilde{\gamma}_n \) converging weakly to zero but, unfortunately, the extension morphism fails to be an isometry.

The assumption that \( \{\gamma_n\}_{n \in I\mathbb{N}} \) is a \( w \)-null basic sequence can be replaced by other conditions which will enable us to proceed as in the previous proof.

Ovsepian and Pelczynski proved (see [LT], for example) that every infinite dimensional separable Banach space verifies that, for each \( \varepsilon > 0 \), there exist two sequences \( \{x_n\}_{n \in I\mathbb{N}} \subset E \) and \( \{\gamma_n\}_{n \in I\mathbb{N}} \subset E' \) such that

(i) \( \gamma_n(x_m) = \delta_{nm} \quad \forall n, m \in I\mathbb{N} \).

(ii) \( \|x_n\| = 1; \quad \|\gamma_n\| \leq 1 + \varepsilon \quad \forall n \in I\mathbb{N} \).

(iii) \( E = \{\{x_n\}_{n \in I\mathbb{N}}\} \).

(iv) \( \{\gamma_n\}_{n \in I\mathbb{N}} \) is a total system over \( E \), i.e. \( \gamma_n(x) = 0 \quad \forall n \in I\mathbb{N} \) implies that \( x = 0 \).

A pair of sequences in these conditions will be called an O-P system. Note that \( \{\gamma_n\}_{n \in I\mathbb{N}} \) is a weak-star null sequence.

Proposition 3.9. Let \( E \) be an infinite dimensional separable Banach space. Let \( \{x_n\}_{n \in I\mathbb{N}} \subset E \) and \( \{\gamma_n\}_{n \in I\mathbb{N}} \subset E' \) be an O-P system. If \( K = \{\gamma_n\}_{n \in I\mathbb{N}} \) then every \( K \)-bounded \( N \)-homogeneous polynomial \( P \) is approximable and extendible by an extension morphism.

Proof. Let \( u : E \rightarrow c_0 \) be as in (5), which is well defined because \( \{\gamma_n\}_{n \in I\mathbb{N}} \) is a weak-star null sequence. Since \( u(x_n) = e_n \), \( \text{Im}(u) \) is a dense subspace of \( c_0 \) and we can define \( Q : c_0 \rightarrow IK \) such that \( Q(u(x)) = P(x) \) for all \( x \in E \). Thus \( P \) is approximable with a representation as in (7). The extension result is derived just as in the previous proposition. ■

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References


